Coupon-coloring and total domination in Hamiltonian planar triangulations

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Abstract

We consider the so-called coupon-coloring of the vertices of a graph where every color appears in every open neighborhood, and our aim is to determine the maximal number of colors in such colorings. In other words, every color class must be a total dominating set in the graph and we study the total domatic number of the graph. We determine this parameter in every maximal outerplanar graph, and show that every Hamiltonian maximal planar graph has domatic number at least two, partially answering the conjecture of Goddard and Henning.

Keywords: coupon-coloring, generalized sun graphs, outerplanar graph, total domatic number, total domination, triangulated disc

1 Introduction

Let G be a graph with no isolated vertex. Cockayne, Dawes and Hedetniemi introduced the concept of the *total domatic number* of graphs in [3]. A subset S of the vertex set $V(G)$ of a graph G is a *total dominating set* if every vertex of G is adjacent to at least one vertex from S. The topic has a large literature, we refer the reader to the excellent surveys of Henning and Yeo [10] and Henning [11] for details on total domination. The maximum number of disjoint total dominating sets is called the *total domatic number*.

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One may see that as a partition of the vertices such that every vertex has a neighbor in each partition class, and the maximal cardinality of partition classes is studied. For that reason, in the forthcoming sections we call a coloring of a graph a *coupon-coloring* if every vertex has neighboring vertices in each color class, for brevity. This name was introduced by Chen, Kim, Tait and Verstraete [2], while Goddard and Henning called such a coloring *thoroughly distributed* [8]. Hence the coupon-coloring number and the total domatic number are equivalent concepts.

The importance of the case when the total domatic (or coupon-coloring) number equals 2 relies on the fact that it is closely related to the so-called Property B of hypergraphs defined by Erdős [7]. More precisely, the neighborhood hypergraph $\Gamma(G)$ of G, consisting of the neighborhoods $N(v)$ of the vertices $v \in V(G)$ as hyperedges, has Property B if and only if G has a coupon-coloring with two colors (i.e., the total domatic number is at least two). In addition, the concept has applications in network science [2], and also related to the panchromatic k-colorings of a hypergraph defined by Kostochka and Woodall [13] concerning vertex colorings with k colors such that every hyperedge contains each color.

We note that it is NP-complete to decide whether the total domatic number is at least two for a given graph, and there are graphs with arbitrarily large minimal degree and total domatic number 1 [16]. On the other hand, the total domatic number of k-regular graphs is of order $\Theta(k/\log k)$ if k is large enough, see [1, 2], and references therein.

The problem of finding the largest possible number of colors in such colorings has been investigated for several graph families [2, 15]. We mainly focus on planar graphs, following the footsteps of Goddard and Henning [8], and also of Dorfling, Hattingh and Jonck [5].

Our paper is organized as follows. In Section 2, we extend a theorem of Goddard and Henning [8] and characterize those maximal outerplanar graphs that have a coupon-coloring with two colors. Section 3 is devoted to the confirmation of their conjecture in a special case, namely we prove that Hamiltonian maximal planar graphs do have a coupon-coloring with two colors. Finally, we discuss some related results and open problems in Section 4.

2 Maximal outerplanar graphs

The total dominating sets of maximal outerplanar graphs have been studied extensively, see [4, 5, 14]. In [8] Goddard and Henning studied the total domatic number for special planar graph families, such as outerplanar graphs.

We will strengthen their result in the forthcoming section.

First we introduce some notations and present some useful observations concerning maximal outerplanar graphs.

Definition 1. The *weak dual* of a maximal outerplanar graph G is the graph that has a vertex for every bounded face of the embedding of G, and an edge for every pair of adjacent bounded faces of G.
The *outer face* (or the boundary of the maximal outerplanar graph) determines a Hamiltonian cycle, and edges not belonging to this cycle are called *chords*. Note that chords correspond to edges in the weak dual.
We can assign a distance for each pair of vertices, which is their minimal ordinary distance on the Hamiltonian cycle, i.e., for a cyclically indexed maximal outerplanar graph of order n, *distance of v_i and v_j or the length of the chord v_iv_j* is $\min\{j-i, n-j+i\}$ if $j > i$; but for convenience, we can allow both of $\{j-i, n-j+i\}$ to represent the distance as we will not consider the induced metric. Vertices are always indexed cyclically.

Observation 2.1. *The weak dual of an n-vertex maximal outerplanar graph G is tree on $n-2$ vertices with maximal degree at most three. If an edge of the weak dual divides the tree into two subtrees of size $|T_1|$ and $|T_2|$, then the chord corresponding to this edge defines distance $|T_1|+1$, or equivalently $|T_2|+1$ in the above sense; i.e., the difference of the indices of corresponding vertices in G is either $|T_1|+1$ or $|T_2|+1$. The maximal number of vertices of degree 2 in G, or equivalently, the maximal number of leaves in the weak dual is at most $n/2$.*

Our key constructions are the so-called *sun graphs* — introduced by Goddard and Henning [8] — and *generalized sun graphs*.

Definition 2. Let G be any maximal outerplanar graph of order $n \geq 3$. The *sun graph* $M(G)$ of G is the graph obtained from G in the following way. Take a new vertex v_e for each edge $e = uw$ on the boundary of G, and join v_e with u and w. The resulting graph will be also a maximal outerplanar graph and has order $2n$.

Alternatively, a sun graph is any maximal planar graph that have half as many vertices of degree 2 as the order of the graph.

Definition 3. A vertex v of a maximal outerplanar graph is called a *central vertex* if no vertex of degree 2 is contained in its closed neighborhood, moreover if we start indexing the vertices along the Hamiltonian cycle (in either direction) from that vertex, then its neighboring vertices have congruent indices modulo 4.

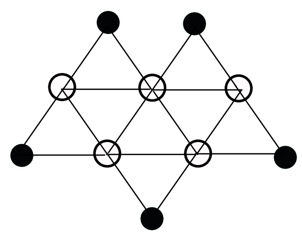

Figure 1: The sun graph $M(G_5)$ of the unique maximal outerplanar graph G_5 of order 5. The vertices of the base graph G_5 are marked with white.

Remark 2.2. *Observe that the existence of a central vertex implies that* $n \equiv 2 \pmod 4$. *Furthermore, central vertices cannot be consecutive on the outer face. Indeed, if* v *and* v^* *were consecutive central vertices, then there would exist a vertex* w *so that* v, v^*, w *form a triangle but then the modulo* 4 *conditions on the distances between* w *and* v *and between* w *and* v^* *would contradict.*

Definition 4. A *generalized sun graph* is a maximal outerplanar graph of order $n \equiv 2 \pmod 4$ that has half as many vertices of degree 2 plus central vertices as the order of the graph.

This directly implies that in a generalized sun graph every second vertex along the outer face is either a central vertex or has degree 2. Also observe the following fact.

Lemma 2.3. *Any chord incident to a central vertex* v_c *in a generalized sun graph* G *partitions* G *into two generalized sun graphs.*

Proof. By the definition and by Remark 2.2, every second vertex is either a vertex of degree 2 or a central vertex in a generalized sun graph. Moreover, central vertices are adjacent to neither vertices of degree 2 nor central vertices. Then it suffices to prove that that two outerplanar graphs divided by a chord e incident to v_c have $4m + 2$ vertices for some $m \geq 1$. This is clear since if $e = v_1 v4l + 2$ (where $v_1 = v_c$) for some $l \geq 1$, then one part has $4l+2$ vertices while the other has $(4k + 2) - (4l + 2) + 2 = 4(k - l) + 2$ vertices where $|V(G)| = 4k + 2$ $(k > l)$. Note that v_c might have degree 2 in the resulting parts. □

The total domination number of maximal outerplanar graphs has been studied by several authors. Dorfling, Hattingh and Jonck [5] showed that, except for two exceptions, every maximal outerplanar graph of order at least 5 has total domination number at most $2n/5$.

This result implies that in general, one cannot obtain a coupon-coloring of a maximal outerplanar graph with more than two colors. Goddard and Henning [8] proved the following theorem.

Theorem 1. *Let n be an integer, $n \geq 4$. If $n \equiv 2 \pmod 4$, then there exists a maximal outerplanar graph of order n without two disjoint total dominating sets. Otherwise (i.e., $n \not\equiv 2 \pmod 4$), every maximal outerplanar graph of order n has two disjoint total dominating sets.*

The evidence for the first part of Theorem 1 relies on the fact that for any maximal outerplanar graph H of order $2k+1$, the corresponding sun graph $M(H)$ of order $4k+2$ admits no disjoint pair of total dominating sets; see [8]. However, this is not the only family of graphs which do not admit a pair of disjoint total dominating sets. We define below a well structured infinite family.

Definition 5. A *parasol graph* P_n with $n = 4k+2$ is defined as follows: Take a k-fan graph with center vertex u_c (i.e., the graph constructed by joining k copies of the triangle graph with a common vertex u_c, also called friendship graph or windmill graph), where $u_c v_i w_i$ denotes a triangle of the k-fan for $1 \leq i \leq k$, and take k copies of the maximal planar graph G_5 where $p_i q_i$ denotes the unique the edge in the ith copy which connects two vertices of degree 3. Identify two vertices v_{i+1} and w_i for each $i \in \{1, 2, \ldots, k-1\}$ and finally two edges $v_j w_j$ and $p_j q_j$ for each $1 \leq i \leq k$ (see Figure 2).

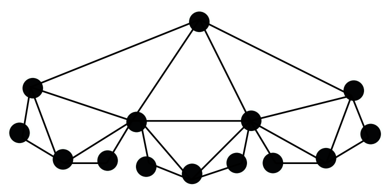

Figure 2: The parasol graph P_{14} on $n = 14$ vertices, a special generalized sun graph

The parasol graphs are examples of generalized sun graphs. A key observation is the following proposition.

Proposition 2.4. *Any generalized sun graph does not admit a pair of disjoint total dominating sets.*

Proof. We show a stronger statement, namely that one cannot color the vertices of generalized sun graphs, using two colors, so that every vertex of degree 2 and central vertex have neighbors from both color classes. Take a minimal counterexample G of order $4k+2$ with a coupon-coloring using two colors. Since the unique generalized sun graph for $k = 1$ clearly does not have a coupon-coloring with two colors, we may suppose $k > 1$.

Index the vertices along the Hamiltonian cycle from v_1 to v_{4k+2}. By definition, we can assume that every vertex of odd index is either a vertex of degree 2 or a central vertex. The cardinality of the vertices implies that there must be two consecutive vertices v_{2i} and v_{2i+2} of even index along the Hamiltonian cycle which have the same color (say white). This would yield a contradiction if v_{2i+1} was a vertex of degree 2, so it must be a central vertex. Since the coloring is a coupon-coloring, G has an edge between a black vertex v_{2j} and the central vertex v_{2i+1}. This edge cuts G into two generalized sun graphs G' (induced by vertices from v_{2i+1} to v_{2j}) and G'' (induced by vertices from v_{2j} to v_{2i+1}) of order less than that of G.

However, G' cannot be properly coupon-colored with two colors by the minimality of G, that is, in every 2-coloring, a vertex of odd index has neighbors from only one color class. Thus this holds for the coloring of G induced by the coloring of G' as well. The vertex violating the condition cannot be v_{2i+1} since it has black and white neighbors. On the other hand, the other vertices of odd index in G' has exactly the same neighborhood as in G, a contradiction. □

Our contribution is the following characterization.

Theorem 2. *Let G be a maximal outerplanar graph of order $n \geq 4$. Then G is either a generalized sun graph or admits two disjoint total dominating sets.*

Before we prove this theorem, we introduce some notations and add some useful observations concerning maximal outerplanar graphs.

Definition 6. Let $G'||G''(x'y', x''y'')$ denote the graph obtained from G' and G'' by identifying their edges $x'y'$ and $x''y''$, respectively.
The operation of *deletion of a (triangular) face $v_iv_jv_k$ $(i < j < k)$* in a maximal outerplanar graph G of order n produces three induced subgraphs by the sets $\{v_i, v_{i+1}, \dots, v_j\}$, $\{v_j, v_{j+1}, \dots, v_k\}$, and $\{v_k, v_{k+1}, \dots, v_i\}$ (written cyclically). If $i + 1 = j$, or $j + 1 = k$, or $k = n$ and $i = 1$, then the corresponding maximal outerplanar subgraph consists of a single edge only, which is a degenerate case.

Note that the deletion of the face f corresponds to the deletion of the corresponding vertex v_f in the weak dual of G.

Lemma 2.5. *([8]) There always exists a chord of length* 3 *or* 4 *in any maximal outerplanar graph* G *of order* $n \geq 6$.

Proof. For $n = 6$, the statement is straightforward. Otherwise consider the weak dual T of G, and delete its leaves. The remaining graph is a tree T' of order at least two. Take a chord corresponding to one of the leaves in T'. Since this leaf in the remaining tree was attached to 1 or 2 leaves of T before the deletion, the chord is of length 3 or 4 via Observation 2.1. $\square$

Lemma 2.6. *In any maximal outerplanar graph* G *of order* $n \geq 5$*, there exists a bounded face such that the deletion of the corresponding vertex in the weak dual graph* T *of* G *creates at most one tree of order greater than* 3 *and at least one of order* 2 *or* 3.

Proof. Since the weak dual T has $n - 2$ vertices, the lemma clearly follows if $n = 5$ or 6 (by deletion of one of its leaves). If $n = 7$, then T must have a vertex of degree 2. Thus the deletion of this vertex creates a subtree of order 2 or 3 and one of order 1 or 2. Suppose that $n = 8$. If T has a vertex of degree 3, its deletion provides three subtrees of which the largest is of size 2 or 3. If T does not have vertices of degree 3 then it is a path and the claim is straightforward.

Finally, suppose that $n \geq 9$.

Label every leaf by L, delete them. Next label the leaves of the remaining tree T' by L', and delete them. Let T'' be the resulting tree. Observe that T'' is not empty since every tree T of maximum degree at most 3 has at most $\frac{1}{2}|V(T)| + 1$ leaves, so

choose a leaf u of T''. We note that there are at most two neighbors of u, and each of them has at most two neighbors with label L. Consequently, the choice u is a desired one. $\square$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. If $n \not\equiv 2 \pmod 4$, then the theorem holds by Theorem 1, where the coupon-coloring is obtained by alternate coloring of pairs along the Hamiltonian cycle on the unique outer face.

Suppose now that $n = 4k + 2$. We will use induction on k and show that if a maximal outerplanar graph G of order $4k + 2$ does not have a coupon-coloring with two colors, then it must be a generalized sun graph.

The case $k = 1$ is easy to check, so suppose that $k \geq 2$. In general, if there exists a chord xy of distance 3, then there exists a 2-coupon-coloring. Indeed, this means that the chord $e = xy$ divides the graph G to a pair of maximal outerplanar graphs of order 4 and $4k$, which can both be colored so that x and also y have the same color in both subgraphs.

If there is no chord defining distance 3, then G has a chord $e = xy$ of distance 4 by Lemma 2.5. This chord cuts down a maximal outerplanar graph G_5 of order 5. Notice that x and y must be vertices of degree 3 in G_5, otherwise we could find a chord of distance 3 in G. Moreover, x and y must have the same color to guarantee that the vertices of degree 2 in G_5 both have a black and a white neighbor.

Consider the triangle face adjacent to e which is not contained in G_5, and denote it by xyz. The deletion of this face of in G yields three maximal outerplanar graph G', G'' and the graph G_5. Note that one of G' and G'' may be degenerate and have only one edge, but the sum of their order is $4k$. Without loss of generality we may assume that $|G'| \leq |G''|$.

Choose the chord of length 4 which has the following property: $|G'|$ is minimal w.r.t. all choices. By Lemma 2.6, we can assume that $|G'| \leq 5$.

Suppose that xz is the edge of G'. We end up with the following four cases depending on the order of G'.

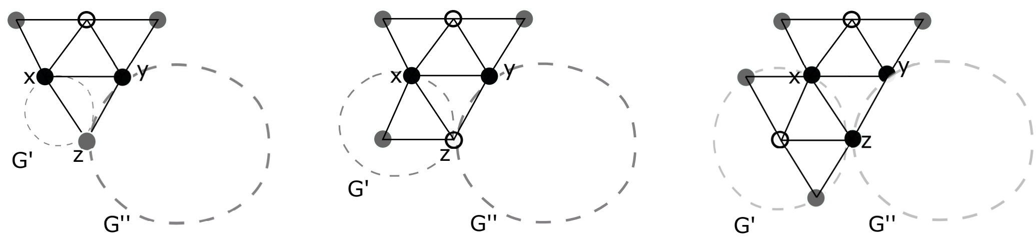

Figure 3: Case 1, Case 2 and Case 4; Gray indicates an undefined color

Case 1 (the left of Figure 3). $|G'| = 2$, the degenerate case. Then G'' has order $4k - 2$. If G'' can be coupon-colored, then z and y both have a black neighbor and white neighbor, and hence, we can easily extend the coupon-coloring to that of G. Otherwise we can use induction, and get that G'' is a generalized sun graph. Hence y or z is either a vertex of degree 2 or a central vertex in G''. Denote this vertex by w_1 and index the vertices of G'' cyclically from w_1 along its outer face.

If $y = w_1$, then we color the vertices in pairs, alternating, along the Hamiltonian cycle of G'' starting from $w_2 = z$, until w_{4k-2} (white-white-black-black...), and color $y = w_1$ black.

We claim that by this way we obtain a coloring where only $w_1 = y$ has monochromatic neighborhood in G''. We only need to verify that w_{4k-2}, colored with white, indeed has a white neighbor. Let w_t be the vertex with the largest index which is a neighbor of $w_1 = y$, except for w_{4k-2}. The given coloring implies that every neighbor of $w_1 = y$ is colored with white since every neighbor is congruent to 2 modulo 4. Since there is no neighbor of

w_1 among $\{w_{t+1}, \dots w_{4k-2}\}$, w_{t+1} and w_{4k-2} must be adjacent, and w_{t+1} is white.

However the graph G_5 will serve as a remedy to coupon-color the whole graph G, since G_5 can be 2-coupon-colored to provide y the missing (black) color.

If $z = w_1$, then it is easy to see that z will be a central vertex in G and G will be a generalized sun graph.

Case 2 (the middle of Figure 3). $|G'| = 3$. In any coupon-coloring with two colors, x and z must have different colors, since they are neighbors of a vertex of degree 2. On the other hand, G_5 implies that x and y must have the same color to provide both colors to the vertices of degree 2. Consequently $G'||G''(zx, zy)$ is 2-coupon-colorable if and only if G is 2-coupon-colorable. On the other hand, $G'||G''(zx, zy)$ is a generalized sun graph if and only if G is a generalized sun graph, hence the claim follows by induction.

Case 3. $|G'| = 4$. We have already solved this case before since xz was a chord of distance 3, which yielded a straightforward coupon-coloring with 2 colors.

Case 4 (the right of Figure 3). $|G'| = 5$. Here xz must correspond to vertices of degree 3 in $|G'|$, otherwise we could find a chord of distance 3. It thus follows that in any 2-coupon-coloring, x and z must have the same color, similarly to the pair x and y. This implies that G has a 2-coupon-coloring if and only if $G'||G''(zx, zy)$ is 2-coupon-colorable. On the other hand, $G'||G''(zx, zy)$ is a generalized sun graph if and only if G is a generalized sun graph, and hence the claim follows by induction. □

3 Hamiltonian maximal planar graphs

A central conjecture of Goddard and Henning is the following.

Conjecture 3.1. ([8]) *Let G be a maximal planar graph (that is, a triangulation) of order at least 4. Then total domatic number of G is at least 2.*

They proved that the conjecture holds if every vertex in G has odd degree. Based on our previous section, we can verify the statement in the Hamiltonian case.

Theorem 3. *Let G be a Hamiltonian maximal planar graph of order at least 4. Then its total domatic number is at least 2.*

We first need the following lemma and corollary.

Lemma 3.2. *In every generalized sun graph of order $4k+2$, the number of chords incident to central vertices is at most $k-1$.*

Proof. For $k=1$, the unique generalized sun graph has no central vertex. In general, choose a chord incident to some central vertex, which has the smallest length. Suppose it is v_iv_j, with central vertex v_i. It divides the outerplanar graph into two generalized sun graphs, induced by the vertices $\{v_i \dots v_j\}$ and $\{v_j \dots v_i\}$ (written cyclically). The condition on the length of v_iv_j implies that one of them cannot contain further diagonals incident to central vertices. However, the other part has cardinality at most $4(k-1)+2$ by the properties of central vertices, hence the statement follows by induction. □

Remark 3.3. *The Parasol graph of order $4k+2$ shows that the bound is sharp.*

Corollary 3.4. *The number of vertices of degree 2 exceeds the number of central vertices in generalized sun graphs. In particular, the number of vertices of degree 2 is at least $k+2$ if the order of the graph is $4k+2$.*

Proof. Let G be a generalized sun graph of order $4k+2$. It has at most $k-1$ central vertices, so it has at least $k+2$ vertices of degree 2. □

Proof of Theorem 3. First notice that any Hamiltonian maximal planar graph can be a graph obtained from two maximal outerplanar graphs by identifying their Hamiltonian cycles. Hence essentially we only have to deal with those maximal planar graphs which are obtained from two generalized sun graphs G_1 and G_2, otherwise the theorem follows from Theorem 2.

Assume that this is the case and the order of the graph is $n=4k+2$. We can exclude the case when the union of the set of vertices of degree 2 and central vertices coincide in the two outerplanar graphs. Indeed, Corollary 3.4 would imply that a pair of vertices of degree 2 also coincide in that case, while every degree must be at least 3 in a maximal planar graph.

We claim that there exists a pair of vertices v_i and v_{i+3} having distance 3 along the Hamiltonian cycle, such that one of them is a vertex of degree 2 in G_1 and the other is a vertex of degree 2 in G_2. Let $\{v_j |\ j \in J\}$ be the set of vertices of degree 2 in the first generalized sun graph G_1 (for some index set J), and let $W := \{v_{j+3} |\ j \in J\}$. It also follows from Corollary 3.4 that W and the set of vertices of degree 2 in the second generalized sun graph G_2 cannot be disjoint, proving our claim.

To end the proof, we define a 2-coupon-coloring of the vertices. Choose a pair of vertices having distance 3 along the Hamiltonian cycle such that one of them is of degree 2 in G_1 and the other is of degree 2 in G_2. Let v_1

and v_4 be such a pair. Color a vertex v_i white if $i \equiv 1, 2 \pmod 4$ and $i > 4$, otherwise color it black. Then it is obvious that every vertex apart from v_2 and v_3 has both black and white neighbors along the Hamiltonian cycle. However, since v_1 and v_4 were vertices of degree 2 in G_1 and G_2 respectively, the edges $v_{4k+2}v_2$ and v_3v_5 are contained in G, ensuring both colors in the neighborhood of v_2 and v_3 as well. □

4 Concluding remarks and open questions

Dorfling, Goddard, Hattingh and Henning studied the minimum number of edges needed to be added to graphs of order n with minimum degree at least two to obtain two disjoint total dominating sets [6]. They proved that at most asymptotically $(n - 2\sqrt{n})$ edges are needed in general to do so. Observe that Theorem 2 implies a similar result concerning maximal outerplanar graphs (which also have minimum degree 2).

Corollary 4.1. *We can obtain a graph having two disjoint total dominating set from every maximal outerplanar graph by the addition of at most one edge.*

Proof. It is enough to check the statement for generalized sun graphs by Theorem 2. Without loss of generality, we may suppose that v_1 is a vertex of degree 2. Then we color the vertices as in the proof Theorem 3, (i.e., color $v_1, \ldots, v_4$ by black and color white-white-black-black ... from v_5). Note that every vertex except v3 has both black and white neighbors in the 2-coloring. Thus, by adding the edge v_3v_5, the 2-coloring is a coupon-coloring of the resulting graph. □

The proof of Theorem 3 and its ingredients suggest that under some weak assumptions the total domatic number of maximal planar graphs exceeds 2. Certain conditions are clearly needed; see Figure 4. This opens the way to the following question.

Problem 4.2. *Characterize maximal planar graphs with total domatic number exactly* 2.

Considering planar graphs, the problem turns out to be rather hard. Koivisto, Laakkonen and Lauri recently proved that it is NP-complete to decide if the vertex set of bipartite planar graph can be partitioned into two total dominating sets (i.e., whether the coupon coloring number is at least 2) [12].

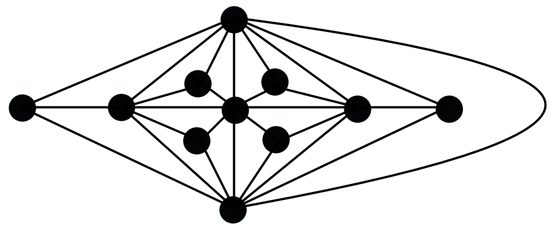

Figure 4: An example for a maximal planar graph having total domatic number 2

Hamiltonicity of maximal planar graphs may serve as an intermediate step towards Conjecture 3.1. Note that a well known result of Whitney shows that every maximal planar graph without separating triangles is Hamiltonian, where a *separating triangle* is a triangle whose removal disconnects the graph. This was even strengthened in the work of Helden who proved that each maximal planar graph with at most five separating triangles is Hamiltonian [9].

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