1. The polynomial $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ with integer non-zero coefficients has n distinct integer roots. Prove that if the roots are pairwise coprime, then a_{n-1} and a_n are coprime.

2. Let P(x) and Q(x) be polynomials with real coefficients such that P(x) = Q(x) for all real values of x. Prove that P(x) = Q(x) for all complex values of x.

3. (a) Determine all polynomials P(x) with real coefficients such that P(x^2) = P^2(x).
(b) Determine all polynomials P(x) with real coefficients such that P(x^2) = P(x)P(x + 1)
(c) Suppose P(x) is a polynomial such that P(x − 1) + P(x + 1) = 2P(x) for all real x. Prove that P(x) has degree at most 1.

4. (USAMO 1975) A polynomial P(x) of degree n satisfies P(k) = k/(k + 1) for k = 0, 1, 2, ..., n. Find P(n + 1).

5. (Brazil 2007) Let P(x) = x^2 + 2007x+ 1. Prove that for every positive integer n, the equation P(P(. . .(P(x)). . .)) = 0 has at least one real solution, where the composition is performed n times.

6. (Russia 2002) Among the polynomials P(x), Q(x), R(x) with real coefficients at least one has degree two and one has degree three. If P^2 (x) + Q^2 (x) = R^2 (x) prove that one of the polynomials of degree three has three real roots.

7. Eisenstein's Criterion: Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in Z[x]$ be a polynomial and p be a prime dividing $a_0, a_1, ..., a_{n-1},$ such that p does not devide an and p^2 does not devide a_0. Then P(x) is irreducible

8. (IMO 1986, #1) Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set {2, 5, 13, d} such that ab − 1 is not a perfect square.

9. Call a positive integer "good" if it can be written in the form a^2 + 3b^2 . Show that the product of two good numbers is good. (Then reverse this construction to show that if 7n is good, then n is good.)

10. Let a, b, c, d be positive integers with ab = cd. Prove that a + b + c + d is composite.

11. Show that 4^n + n^4 is composite for all integers n ≥ 2.

12. Show that 19^{19} cannot be written as $m^4 + n^3$ for any integers m and n.

13. Recall that e is given by the infinite sum e = 1 + 1/1! + 1/2! + 1/3! + · · · . Show that e is irrational.

14. (a) (USAMO 1974) Let a, b, c be three distinct integers, and let P be a polynomial with integer coefficients. Show that in this case the conditions P(a) = b, P(b) = c, P(c) = a cannot be satisfied simultaneously.
(b) Let P(x) be a polynomial with integer coefficients, and let n be an odd positive integer. Suppose that $x_1, x_2, \ldots, x_n$ is a sequence of integers such that $x_2 = P(x_1)$, $x_3 = P(x_2)$, . . . , x_n =

$P(x_{n-1})$, and $x_1 = P(x_n)$. Prove that all the x_i 's are equal.
(c) (Putnam 2000) Let f(x) be a polynomial with integer coefficients. Define a sequence a_0, a_1, . . . of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.
(d) (IMO 2006) Let P(x) be a polynomial of degree n > 1 with integer coefficients and let k be a positive integer. Consider the polynomial Q(x) = P(P(. . .(P(x). . .)) (k times P) Prove that there are at most n integers t such that Q(t) = t.

15. Let a, b, c be nonzero integers such that both a/b + b/c + c/a and a/c + c/b + b/a are integers. Prove that |a| = |b| = |c|.

16. (China 1995) Alice and Bob play a game with a polynomial of degree at least 4: $x^{2n} + @x^{2n-1} + @x^{2n-2} + \cdots + @x + 1$. They fill in real numbers to empty boxes(@) in turn. If the resulting polynomial has no real root, Alice wins; otherwise, Bob wins. If Alice goes first, who has a winning strategy?

17. (USAMO 2002) Prove that any monic polynomial of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

18. Let n ≥ 3 be a positive integer. n lilypads are placed in a circle, with one lilypad marked start. There are n frogs, labeled 1, · · · , n, such that frog k in one leap jumps k lilypads in the clockwise direction. For each k ∈ {1, · · · , n}, if frog k is at start, how many (positive number of) leaps does the frog make to return to start for the first time?

19. Let a, n, d be positive integers such that gcd(a, n) = 1. Let m = ordn(a). Express $ord_n(a^d)$ in terms of m and d

20. Let p ≥ 5 be a prime and 1 ≤ n < p − 1 a positive integer. Prove that $1^n + 2^n + \cdots + (p-1)^n$ is divisible by p.

21. Let p be an odd prime and g a primitive root modulo p. Prove that $g^{(p-1)/2} \equiv -1 \pmod p$.

22. Prove that there are no positive integer solutions to $4ab - a - b = c^2$.

23. (Putnam, May 1977) Determine all solutions of the system x+y+z=w and 1/x+1/y+1/z=1/w .

24. (USSR Olympiad) Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every positive integer n.

25. (Po, 2004) Prove that $x^4 - x^3 - 3x^2 + 5x + 1$ is irreducible.

26. (Canadian Olympiad, 1970) Let P(x) be a polynomial with integral coefficients. Suppose there exist four distinct integers a, b, c, d with P(a) = P(b) = P(c) = P(d) = 5. Prove that there is no integer k with P(k) = 8.

27. (Elgin, MOP 1997) For which n is the polynomial $1 + x^2 + x^4 + \cdots + x^{2n-2}$ divisible by the polynomial $1 + x + x^2 + \cdots + x^{n-1}$?

28. (Hungarian Olympiad, 1981) Show that there is only one natural number n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

29. Find all positive integers x such that $2^m + 3n = m! + n^2 + 4$.

30. Find all positive integers k such that $k + 4 \mid k^3 + k^2 + 2k + 1$.

31. Find all integers k such that $k^2 + k + 1 \mid 3k^2 + k + 4$.

32. Find all positive integers a, b, c such that $a \mid b + c$, $b \mid a + c$, $c \mid a + b$.

33. Find all positive integers a, b such that $a^2 = b^2 + 3b$.

34. Find all positive integers x, y such that $x^3 = y^3 + 2y^2 + 8$.

35. Find all positive integers n such that $n^2 + 1 \mid 5n + 1$.

36. Find all positive integers a, b such that $a^4 + b^2 + 2 = 2ab$

37. Find all integers x, y such that $5/x + 3/y = 1$

38. Find all primes p, q such that $p^2 - 2q^2 = 1$.

39. Find all integers a, b such that a^2+4b and b^2+4a are both perfect squares.

40. Find all integers a, b, c, d such that $a^2 + b^2 = 7(c^2 + d^2)$.

41. Find all primes p, q, r such that $p(q - r) = q + r$.

42. Find all positive integers m, n such that $1 + 5 \cdot 2^m = n^2$.

43. Find all positive integers x, y such that $x(x + 1)(x + 2)(x + 3) = y^2$.

44. Find all integers x, y such that $x^3 = y^3 + 2y^2 + 1$.

45. Find all positive integers m, n such that $m! = n^3 + 3$.